$$= \quad 2 - 2\left(a^2 + b^2 + c^2\right).$$

Now

$$a^2 + b^2 + c^2 = \left(a - \frac{1}{3}\right)^2 + \left(b - \frac{1}{3}\right)^2 + \left(c - \frac{1}{3}\right)^2 + \frac{2(a+b+c)}{3} - \frac{1}{3} \geq \frac{1}{3},$$

so that $1 + a^2 + b^2 + c^2 \geq 2 - 2\left(a^2 + b^2 + c^2\right)$.

This proves (1) and completes the solution.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

By the AM-GM inequality,

$$\begin{aligned}
\frac{1+a}{bc} + \frac{1+b}{ca} + \frac{1+c}{ab} &= \frac{a + a^2 + b + b^2 + c + c^2}{abc} \\
&= \frac{1 + a^2 + b^2 + c^2}{abc} \\
&= \frac{(a+b+c)^2 + a^2 + b^2 + c^2}{abc} \\
&= \frac{(2a^2 + 2bc) + (2b^2 + 2ca) + (2c^2 + 2ab)}{abc} \\
&\geq 4\frac{a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}}{abc} \\
&= \frac{4}{\sqrt{bc}} + \frac{4}{\sqrt{ca}} + \frac{4}{\sqrt{ab}}.
\end{aligned}$$

The conclusion follows since

$$\frac{1}{\sqrt{xy}} \geq \frac{1}{\sqrt{x^2 + y^2 - xy}}.$$

(Note that this inequality is equivalent to $x^2 + y^2 - xy \geq xy$ which is obviously true.)

Also solved by Arkady Alt, San Jose, CA; Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; David E. Manes, Oneonta, NY; Titu Zvonaru, Comănesti, Romania jointly with Neculai Stanciu, Buzău, Romania, and the proposer.

- **5181:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Calculate:
  $$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{n \cdot m}{(n+m)!}.$$

Solution 1 by Anastasios Kotronis, Athens, Greece The summands being all positive we can sum by triangles :

$$\begin{aligned}
\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{nm}{(n+m)!} &= \sum_{k,\ell,n\in\wedge k+\ell=n}\frac{nm}{(n+m)!}=\sum_{n=2}^{+\infty}\frac{\sum_{\ell=1}^{n-1}(n-\ell)\ell}{n!}\\
&= \frac{1}{6}\sum_{n=2}^{+\infty}\frac{(n-1)n(n+1)}{n!}=\frac{1}{6}\sum_{n=2}^{+\infty}\frac{(n+1)}{(n-2)!}\\
&= \frac{1}{6}\sum_{n=0}^{+\infty}\frac{(n+3)}{n!}=\frac{1}{6}\sum_{n=0}^{+\infty}\frac{1}{n!}\frac{dx^{n+3}}{dx}\Big|_{x=1}\\
&= \frac{1}{6}\frac{d}{dx}\left(\sum_{n=0}^{+\infty}\frac{x^{n+3}}{n!}\right)\Big|_{x=1}=\frac{1}{6}\frac{d\left(x^3e^x\right)}{dx}\Big|_{x=1}\\
&= \frac{2e}{3}.
\end{aligned}$$

Solution 2 by Arkady Alt, San Jose, CA

Let $k=m+n$. Then $m=k-n$ and domain of summation $\begin{cases}1\le n\\1\le m\end{cases}$ can be represented as $\begin{cases}2\le k\\1\le n\le k-1\\m=k-n\end{cases}$. Hence,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{nm}{(n+m)!}=\sum_{k=2}^{\infty}\sum_{n=1}^{k-1}\frac{n\,(k-n)}{k!}=\sum_{k=2}^{\infty}\frac{1}{k!}\sum_{n=1}^{k-1}n\,(k-n)=\sum_{k=2}^{\infty}\frac{1}{k!}\sum_{n=1}^{k-1}n\,(k-n)\,.$$

Since

$$\begin{aligned}
\sum_{n=1}^{k-1}n\,(k-n) &= \frac{k^2\,(k-1)}{2}-\frac{(k-1)\,k\,(2k-1)}{6}\\
&= \frac{k}{6}\left(3k^2-3k-2k^2+3k-1\right)\\
&= \frac{k\,(k^2-1)}{6},
\end{aligned}$$

then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{nm}{(n+m)!} = \frac{1}{6}\sum_{k=2}^{\infty}\frac{k+1}{(k-2)!}$$

$$
\begin{aligned}
&= \frac{1}{6}\sum_{k=0}^{\infty}\frac{k+3}{k!} \\
&= \frac{1}{6}\left(\sum_{k=0}^{\infty}\frac{3}{k!}+\frac{k}{k!}\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k!}+\frac{1}{6}\sum_{k=1}^{\infty}\frac{1}{(k-1)!} \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{k!}+\frac{1}{6}\sum_{k=0}^{\infty}\frac{1}{k!} \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{2}+\frac{1}{6}\right) \\
&= \frac{2}{3}\sum_{k=0}^{\infty}\frac{1}{k!} \\
&= \frac{2e}{3}.
\end{aligned}
$$

Solution 3 by the proposer

The series equals $\frac{2e}{3}$. First we note that for $m \geq 0$ and $n \geq 1$ one has that

$$\int_0^1 x^m(1-x)^{n-1}dx = B(m+1,n) = \frac{m!\cdot(n-1)!}{(n+m)!}.$$

Thus,

$$
\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n\cdot m}{(n+m)!} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n}{(n-1)!}\cdot\frac{1}{(m-1)!}\int_0^1 x^m(1-x)^{n-1}dx \\
&= \int_0^1\left(\sum_{n=1}^{\infty}\frac{n}{(n-1)!}(1-x)^{n-1}\right)\cdot\left(\sum_{m=1}^{\infty}\frac{x^m}{(m-1)!}\right)dx \\
&= \int_0^1\left(1+\sum_{n=2}^{\infty}\frac{n}{(n-1)!}(1-x)^{n-1}\right)\cdot xe^x dx \\
&= \int_0^1\left(1+\sum_{n=2}^{\infty}\frac{(1-x)^{n-1}}{(n-2)!}+\sum_{n=2}^{\infty}\frac{(1-x)^{n-1}}{(n-1)!}\right)\cdot xe^x dx \\
&= \int_0^1\left(1+(1-x)e^{1-x}+e^{1-x}-1\right)\cdot xe^x dx \\
&= e\int_0^1(2-x)x dx = \frac{2e}{3},
\end{aligned}
$$

and the problem is solved.

Also solved by Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, "Tor Vergata" University, Rome, Italy, and Albert Stadler, Herrliberg, Switzerland.